

ON A CLASS OF NON-INTEGRABLE MULTIPLIERS FOR THE JACOBI TRANSFORM

TROELS ROUSSAU JOHANSEN

ABSTRACT. We show that a bounded function m on \mathbb{R} not necessarily integrable at infinity may still yield L^p -bounded convolution operators for the Jacobi transform if the nontangential boundary values of $\omega \cdot m$ along the edges of a certain strip in \mathbb{C} yield Euclidean Fourier multipliers, where ω is a function of the form $\omega(\lambda) = (\lambda^2 + 4\rho^2)^{\alpha+1/4}$. This partially generalizes similar results by Giulini, Mauceri, and Meda (on rank one symmetric spaces) and Astengo (on Damek–Ricci spaces).

1. INTRODUCTION AND STATEMENT OF RESULT

The study of translation invariant operators has played a decisive role in the development of Euclidean harmonic analysis, as evidenced, for example, by the landmark paper [13] by Hörmander. A close connection between said translation invariant operators, the Fourier transform, and distributions was uncovered, as such operators turned out to be Fourier multiplier operators, or, what amounts to the same thing, convolution operators with suitable kernels. It didn't take long for the experts to seek new venues for their inquiries. One of the first was the important paper [5], where exciting non-Euclidean phenomena were uncovered, in the setting of noncommutative harmonic analysis on a noncompact symmetric space.

Let us specialize to the rank one situation for the moment and define $\Omega_p = \{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| < |2/p - 1|\rho\}$, where ρ is a certain constant associated with the symmetric space G/K (half the sum of positive roots, see Section 2). Clerc and Stein observed that an L^p -multiplier for the so-called spherical transform of a Riemannian symmetric space of the noncompact type always has a holomorphic extension to the strip Ω_p . Several multiplier results followed the publication of [5] and while we cannot adequately recount the complete literature, let us at least mention [22] (the rank one case) and [1] for the general rank case (where the strip Ω_p is replaced by a tube domain T_p over a certain cone in the dual of the Lie algebra of the Iwasawa-group A in G). More recent advances include [14], as well as [4]. The latter establishes the results from [1] in the context of Chébli–Trimèche hypergroups (which subsumes the spherical analysis on a rank one symmetric space, and more generally the Jacobi analysis we are dealing with).

It is well-known that an L^p -multiplier m for the spherical transform on G/K is determined by its boundary value on the edge on Ω_p , and Anker showed that if this boundary value satisfies a Mihlin–Hörmander condition of sufficiently high order, then the function is an L^p -multiplier. A multiplier result with less restrictive assumptions on the multiplier was obtained in [11] (and generalized to Damek–Ricci spaces in [3]), and it is the purpose of the present paper to establish a ‘spherical’ counterpart to both papers in the context of Jacobi analysis. Giulini et.al. observed that there exists a function ω , holomorphic and non-vanishing in a neighborhood of

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Ω_1 such that m is still a multiplier if merely the nontangential boundary value of ωm satisfies Mihlin–Hörmander conditions, so a class of multipliers *larger* than the one considered by Anker is thereby allowed. Additional remarks are to be found in the Introduction and Section 2 of [11]. In essence m is allowed to be less regular at infinity, in particular be non-integrable. This extension was not investigated in [4] but our results generalize to that setting.

The precise formulation is as follows. Let $\omega(\lambda) = (\lambda^2 + 4\rho^2)^{\alpha+1/4}$.

Theorem 1.1. *Let m be an even, holomorphic function on Ω_1 . If ωm is bounded on Ω_1 and its nontangential boundary value $(\omega m)_\rho$ at the upper boundary line $\{\lambda + i\rho\}$ of Ω_1 belongs to $\mathcal{M}_p(\mathbb{R})$ for some $p \in (1, \infty)$, then m is an L^p -multiplier for the Jacobi transform, and there exists a finite constant c such that $\|m\|_{\mathcal{M}_p} \leq c\|(\omega m)_\rho\|_{\mathcal{M}_p(\mathbb{R})}$.*

Here we adhere to the following notation and terminology: Denote by $\mathcal{M}_p(\mathbb{R})$ the space of Euclidean multipliers and by \mathcal{M}_p the space of Jacobi multipliers. The *multiplier norm* of a function m is by convention the operator norm of $f \mapsto \mathcal{F}^{-1}(\mathcal{F}f \cdot m)$ acting on $L^p(\mathbb{R})$, and similarly for the Jacobi multipliers. These choices of norm turn $\mathcal{M}_p(\mathbb{R})$ and \mathcal{M}_p into Banach spaces. Let $d\mu(t) = (2 \sinh t)^{2\alpha+1} (2 \cosh t)^{2\beta+1} dt$ (the significance of this measure is explained in Section 2) and denote by $\mathcal{CO}_p^q(d\mu)$ the space of all linear operators that map boundedly from $L^p(d\mu)$ to $L^q(d\mu)$ and commute with (left) translation. The relevant translation is introduced in Equation (3) below. We write \mathcal{CO}_p instead of $\mathcal{CO}_p^p(d\mu)$, whereas the Euclidean analogue shall always be denoted by $\mathcal{CO}_p(\mathbb{R})$. It is standard that every operator $T \in \mathcal{CO}_p^p(d\mu)$ has the form $Tf = k \star f$ for a unique, suitable function k , and where \star is a suitable convolution (see Equation (4)). By a slight abuse of terminology we say that a function k belongs to $\mathcal{CO}_p^q(d\mu)$ if the associated convolution operator $f \mapsto k \star f$ is $L^p - L^q$ bounded, hence in $\mathcal{CO}_p^q(d\mu)$.

The proof will follow closely the approach in [11] and [3] with one crucial difference (and several smaller technical ones). We cannot use the Herz restriction principle, as we do not have any natural subgroups to which we restrict multipliers. In the present setup transference is the proper replacement, as was also utilized in both [22] and [16]. The transference result is from [10] and it must be pointed out that the proof of the transference theorem is much more difficult than the version used in [22], where group-invariance of the convolution kernel may be exploited. An important realization is that the use of the Figà-Talamanca–Herz algebra $A_p(\mathbb{R})$ in [11] is still permissible in the Jacobi setting, once we have transferred the analysis of the Jacobi multipliers to an Euclidean setting. We refer the reader to [7] for details on the Figà-Talamanca–Herz algebra $A_p(\mathbb{R})$ as well as the Herz restriction principle, and to [6] for further details on transference.

Of a more technical level, we mention new \mathbf{c} -function estimates (necessitated by α, β not being half-integers), the details are summarized in Lemma 2.1. Estimates involving the density $\Delta(t)$ also tend to become more complicated.

A word on notation: Error terms are always denoted by E or e , sometimes with indices, like E_1 and $E_{1,1}$. This is not to imply that the different terms are somehow related, rather it is a matter of notational convenience. The notation $a \lesssim b$ is used as shorthand for an estimate of the form $a \leq cb$ for some constant c ; this constant c might change from line to line. We write out the actual constants if they are important for the conclusion.

2. JACOBI ANALYSIS

In this section we briefly collect the pertinent definitions and facts relevant for Jacobi analysis. A much more detailed account can be found in [18], for example. Let $(a)_0 = 1$ and

$(a)_k = a(a+1)\cdots(a+k-1)$. The hypergeometric function ${}_2F_1(a, b; c, z)$ is defined by

$${}_2F_1(a, b; c, z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k, \quad |z| < 1;$$

the function $z \mapsto {}_2F_1(a, b; c, z)$ is the unique solution of the differential equation

$$z(1-z)u''(z) + (c - (a+b+1)z)u'(z) - abu(z) = 0$$

which is regular in 0 and equals 1 there. The Jacobi function with parameters (α, β) (which will assumed to be *real*) is defined by $\varphi_{\lambda}^{(\alpha, \beta)}(t) = {}_2F_1(\frac{1}{2}(\alpha + \beta + 1 - i\lambda), \frac{1}{2}(\alpha + \beta + 1 + i\lambda); \alpha + 1, -\sinh^2 t)$. For $|\beta| < \alpha + 1$, the system $\{\varphi_{\lambda}^{(\alpha, \beta)}\}_{\lambda \geq 0}$ is a continuous orthonormal system in \mathbb{R}_+ with respect to the weight $\Delta_{\alpha, \beta}(t) = (2 \sinh t)^{2\alpha+1} (2 \cosh t)^{2\beta+1}$, $t > 0$. Assume that $\alpha \neq -1, -2, \dots$, $\alpha > \frac{1}{2}$, and $\alpha > \beta > -\frac{1}{2}$. The Jacobi-Laplacian is the operator $\mathcal{L} = \mathcal{L}_{\alpha, \beta} = \frac{d^2}{dt^2} + ((2\alpha + 1) \coth t + (2\beta + 1) \tanh t) \frac{d}{dt}$, by means of which the Jacobi function $\varphi_{\lambda}^{(\alpha, \beta)}$ may alternatively be characterized as the unique solution to

$$(1) \quad \mathcal{L}_{\alpha, \beta} \varphi + (\lambda^2 + \rho^2) \varphi = 0$$

on \mathbb{R}_+ satisfying $\varphi_{\lambda}(0) = 1$ and $\varphi'_{\lambda}(0) = 0$. It is thereby clear that $\lambda \mapsto \varphi_{\lambda}(t)$ is analytic for all $t \geq 0$. Moreover, for $\text{Im } \lambda \geq 0$, there exists a unique solution ϕ_{λ} to the same equation satisfying $\phi_{\lambda}(t) = e^{(i\lambda - \rho)t}(1 + o(1))$ as $t \rightarrow \infty$, and $\lambda \mapsto \phi_{\lambda}(t)$ is therefore also analytic for $t \geq 0$.

In analogy with the case of symmetric spaces, one proceeds to show the existence of a function $\mathbf{c} = \mathbf{c}_{\alpha, \beta}$ for which $\varphi_{\lambda}(t) = \mathbf{c}(\lambda)e^{(i\lambda - \rho)t}\phi_{\lambda}(t) + \mathbf{c}(-\lambda)e^{(-i\lambda - \rho)t}\phi_{-\lambda}(t)$. Since we adhere to the conventions and normalization used in [9], the \mathbf{c} -function is given by

$$\mathbf{c}(\lambda) = \frac{2^{\rho - i\lambda} \Gamma(i\lambda) \Gamma(\alpha + 1)}{\Gamma(\frac{1}{2}(\rho + i\lambda)) \Gamma(\frac{1}{2}(\rho + i\lambda) - \beta)}.$$

Observe that for $\alpha, \beta \neq -1, -2, \dots$, $\mathbf{c}(-\lambda)^{-1}$ has finitely many poles for $\text{Im } \lambda < 0$ and none if $\text{Im } \lambda \geq 0$ and $\text{Re } \rho > 0$. It follows from Stirling's formula that for every $r > 0$ there exists a positive constant c_r such that

$$(2) \quad |\mathbf{c}(-\lambda)|^{-1} \leq c_r (1 + |\lambda|)^{\text{Re } \alpha + \frac{1}{2}} \text{ if } \text{Im } \lambda \geq 0 \text{ and } \mathbf{c}(-\lambda') \neq 0 \text{ for } |\lambda' - \lambda| \leq r.$$

Lemma 2.1. Assume $\alpha > \beta > -\frac{1}{2}$.

- (i) For every integer M there exist constants $c_i, i = 0, \dots, M-1$ (depending on α, β , and M) such that

$$|\mathbf{c}(\lambda)|^{-2} \sim c_0 |\lambda|^{2\alpha+1} \left\{ 1 + \sum_{j=1}^{M-1} c_j \lambda^{-j} + O(\lambda^{-M}) \right\} \text{ as } |\lambda| \rightarrow \infty.$$

- (ii) Let $\mathbf{d}(\lambda) = |\mathbf{c}(\lambda)|^{-2}$, $\lambda \geq 0$, and $k \in \mathbb{N}_0$. There exists a constant $c_k = c_{k, \alpha, \beta}$ such that

$$\left| \frac{d^k}{d\lambda^k} \mathbf{d}(\lambda) \right| \leq c_k (1 + |\lambda|)^{2\alpha+1-k}.$$

- (iii) $\mathbf{c}'(\lambda) \sim \mathbf{c}(\lambda)O(\lambda^{-1})$ and $\mathbf{c}''(\lambda) \sim \mathbf{c}(\lambda)O(\lambda^{-2})$.

In particular $|\frac{d}{d\lambda} \mathbf{c}(\lambda)^{-1}| = |\mathbf{c}(\lambda)^{-2} \mathbf{c}'(\lambda)| \lesssim |\mathbf{c}(\lambda)^{-1} \frac{1}{\lambda}| \lesssim |\lambda|^{\alpha - \frac{1}{2}}$ for $|\lambda|$ large.

Proof. We refer the reader to [15, Lemma 2.1] for a proof. This improves on the usual asymptotic statement that $|\mathbf{c}(\lambda)|^{-2} \sim |\lambda|^{2\alpha+1}$ as $|\lambda| \rightarrow \infty$, cf. [22, Lemma 4.2]. \square

Let $d\nu(\lambda) = d\nu_{\alpha,\beta}(\lambda) = (2\pi)^{-\frac{1}{2}} |\mathbf{c}(\lambda)|^{-2} d\lambda$ and denote by $L^p(d\nu)$ the associated weighted Lebesgue space on \mathbb{R}_+ ; note that $\mathbf{c}(\lambda)\mathbf{c}(-\lambda) = \mathbf{c}(\lambda)\overline{\mathbf{c}(\lambda)} = |\mathbf{c}(\lambda)|^2$ whenever $\alpha, \beta, \lambda \in \mathbb{R}$. The Jacobi transform, initially defined for $f \in C_c^\infty(\mathbb{R}_+)$ by

$$\widehat{f}(\lambda) = \frac{\sqrt{\pi}}{\Gamma(\alpha+1)} \int_0^\infty f(t) \varphi_\lambda(t) d\mu(t),$$

extends to a unitary isomorphism from $L^2(d\mu)$ onto $L^2(d\nu)$, and the inversion formula is the statement that

$$f(t) = \int_0^\infty \widehat{f}(\lambda) \varphi_\lambda(t) d\nu(\lambda)$$

holds in the L^2 -sense, cf. [17, Formula 4.5]. The limiting case $\alpha = \beta = -\frac{1}{2}$ is the Fourier-cosine transform, which we will not study. One easily verifies that $\widehat{\mathcal{L}f}(\lambda) = -(\lambda^2 + \rho^2)\widehat{f}(\lambda)$.

Remark 2.2. For special values of α and β , determined by the root system of a rank one Riemannian symmetric space, the functions φ_λ are the usual spherical functions of Harish-Chandra. To be more precise assume G/K is a rank one Riemannian symmetric space of noncompact type, with positive roots α and 2α . Furthermore let p denote the multiplicity of α and q the multiplicity of 2α (we allow q to be zero). With $\alpha := \frac{1}{2}(p+q-1)$ and $\beta := \frac{1}{2}(q-1)$ both real, and $p = 2(\alpha-\beta)$ and $q = 2\beta+1$, the function $\varphi_\lambda^{(\alpha,\beta)}$ is precisely the usual elementary spherical function φ_λ as considered by Harish-Chandra, $\rho = \alpha + \beta + 1 = \frac{1}{2}(p+2q)$ as it should be, and $\dim(G/K) = p+q+1 = 2\alpha+2$.

A similar choice of parameters α, β reveals that even spherical analysis on Damek–Ricci spaces is subsumed by the present setup. This was exploited in [2]. One should also observe that Jacobi analysis can (perhaps should) be placed in the framework of harmonic analysis of hypergeometric functions associated to root systems; according to [21, p. 89f], the hypergeometric functions for a rank one root system with non-negative multiplicity function k (the construction of which is explained, for example, in [20]) are then expressed by

$$F(\lambda, k, t) := {}_2F_1\left(\frac{\lambda+\rho}{2}, \frac{-\lambda+\rho}{2}, k_1+k_2+\frac{1}{2}, -\sinh^2 t\right).$$

These are special types of Jacobi functions; with $\alpha = k_1+k_2-\frac{1}{2}$, and $\beta = k_2-\frac{1}{2}$, one observes that $F(i\lambda, k; t) = \varphi_\lambda^{(\alpha,\beta)}(t)$. The ideal situation where $\alpha > \frac{1}{2}$, $\alpha > \beta > -\frac{1}{2}$ thus amounts to the requirement that $k_2 > 0$ and $k_1 > 1-k_2$.

Recall from [9, Formula (5.1)] the generalized translation τ_x of a suitable function f on \mathbb{R}_+ , which is defined by

$$(3) \quad (\tau_x f)(y) = \int_0^\infty f(z) K(x, y, z) d\mu(z)$$

where K is an explicitly known kernel function such that

$$\varphi_\lambda(x)\varphi_\lambda(y) = \int_0^\infty \varphi_\lambda(z) K(x, y, z) d\mu(z).$$

In fact (cf. [9, Formulae (4.16),(4.19)]), for $|s - t| < u < s + t$,

$$\begin{aligned} K(s, t, u) &= \frac{c_{\alpha, \beta}}{(\sinh s \sinh t \sinh u)^{2\alpha}} \int_0^\pi (1 - \cosh^2 s - \cosh^2 t - \cosh^2 u \\ &\quad + 2 \cosh s \cosh t \cosh u \cosh y)_+^{\alpha - \beta - 1} \sin^{2\beta} y \, dy \\ &= \frac{2^{\frac{1}{2} - \rho} \Gamma(\alpha + 1) (\cosh s \cosh t \cosh u)^{\alpha - \beta - 1}}{\Gamma(\alpha + \frac{1}{2}) (\sinh s \sinh t \sinh u)^{2\alpha}} \\ &\quad \times (1 - B^2)^{\alpha - \frac{1}{2}} {}_2F_1(\alpha + \beta, \alpha - \beta; \alpha + \frac{1}{2}; \frac{1}{2}(1 - B)) \end{aligned}$$

where $B(s, t, u) = \frac{\cosh^2 s + \cosh^2 t + \cosh^2 u - 1}{2 \cosh s \cosh t \cosh u}$; elsewhere $K \equiv 0$. The associated generalized convolution product of two functions $f, g \in L^2(d\mu)$ is defined by

$$(4) \quad f \star g(x) = \int_0^\infty f(y) (\tau_x g)(y) \, d\mu(y) = \int_0^\infty f(y) g(z) K(x, y, z) \, d\mu(z) \, d\mu(y).$$

This convolution is associative and distributive, and by [9, Equation (5.4)(iv)], $\widehat{f \star g}(\lambda) = \widehat{f}(\lambda) \widehat{g}(\lambda)$. The usual inequalities for convolutions continue to hold, as we have the following general form of the Young inequality.

Proposition 2.3. *Let p, q , and r be such that $1 \leq p, q, r \leq \infty$ and $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}$. The convolution $f \star g$ of $f \in L^p(d\mu)$ and $g \in L^q(d\mu)$ is then well-defined as a function in $L^r(d\mu)$, and $\|f \star g\|_r \leq \|f\|_p \|g\|_q$.*

Proof. See [9, Theorem 5.4]. □

Definition 2.4. Let m be a bounded, measurable, even function on \mathbb{R} , and let T_m be the bounded linear operator defined for $f \in L^2(d\mu)$ by $\widehat{T_m f}(\lambda) = m(\lambda) \widehat{f}(\lambda)$, $\lambda \in \mathbb{R}$. The function m is called an L^p -multiplier for the Jacobi transform, with $p \in (1, \infty)$, if the operator T_m extends from $L^2(d\mu) \cap L^p(d\mu)$ to a bounded linear operator on $L^p(\mathbb{R}_+, d\mu)$.

Remark 2.5. The multiplier results in [3] and [11] are formulated for operators acting on functions that are not necessarily radial. The analogy in Jacobi analysis would be to consider functions on \mathbb{R} that are not necessarily even, and our main theorem can be reformulated accordingly as follows. Write a function f on \mathbb{R} as the sum of its even and odd parts, $f = f_e + f_o$, and notice that one can still define the convolution between an even and an odd function. One verifies that $K(-x, y, z) = (-1)^{2\alpha} K(x, y, z)$ for all $x, y, z > 0$, so that $|K(-x, y, z)| = |K(x, y, z)|$ and correspondingly $|K(x, y, z) f(x)|^p \leq 2^p |K(x, y, z) f_e(x)|^p$. The norm of $k \star f$ (k still even) as an element of $L^p(\mathbb{R}, d\mu)$ is therefore controlled by the norm of $k \star f_e$, which is in $L^p(\mathbb{R}_+, d\mu)$. While this extension is straightforward, it is also cumbersome to write all the time. All statements to follow can be modified to be about $L^p(\mathbb{R}, d\mu)$ rather than $L^p(\mathbb{R}_+, d\mu)$ but since one cannot naturally identify $L^p(\mathbb{R})$ with $L^p(G/K)$ in the case of α, β being geometric, we do not obtain statements about operators acting on $L^p(G/K)$. While going from $L^p(\mathbb{R}_+)$ to $L^p(\mathbb{R})$ in Jacobi analysis is straightforward, the same cannot be said about $L^p(K \setminus G/K)$ versus $L^p(G/K)$.

3. LOCAL ANALYSIS

We prove Theorem 1.1 by separately investigating the local and the global part of the kernel. Fix a smooth, even function ψ on \mathbb{R} such that $0 \leq \psi \leq 1$, $\psi(t) \equiv 1$ for $|t| \leq R_0^{1/2}$, and $\psi(t) \equiv 0$

for $|t| \geq R_0$, where R_0 is the constant from [15, Lemma 3.1]¹, see also [22, Theorem 2.1]. Let k be the inverse Jacobi transform of the multiplier function m , regarded as an even distribution on \mathbb{R} . As in [11], we cancel out possible poles by introducing the modified multiplier function $M(\lambda) := m(\lambda)\mathbf{c}(-\lambda)^{-1}$, $\lambda \in \mathbb{R}$. Since M extends to a function that is holomorphic in Ω_1 and bounded on strips of the form $\{z \in \mathbb{C} : \varepsilon - \rho \leq \operatorname{Im} z < \rho\}$, $\varepsilon > 0$, the Fatou lemma guarantees that M has a nontangential limit M_ρ at almost every point of the line $\{\lambda + i\rho : \lambda \in \mathbb{R}\}$.

Proposition 3.1. *Let m be an even function on \mathbb{R} with the property that M belongs to $\mathcal{M}_p(\mathbb{R})$ for some $p \in (1, \infty)$, and let $k = m^\vee$. Then $\psi k \in \mathcal{CO}_p$ and $\|\psi k\|_{\mathcal{CO}_p} \lesssim \|M\|_{\mathcal{M}_p(\mathbb{R})}$.*

Remark 3.2. In what follows we will assume without loss of generality that the multiplier function m be rapidly decreasing. The reduction to this special situation is based on a standard use of heat kernel techniques, already indicated in [22, Remark 1, p.266] and made more precise in the proof of [3, Proposition 4.3]. Let us briefly recall the technique.

Let m be an arbitrary bounded measurable function on \mathbb{R} and define $m_t(\lambda) = m(\lambda)e^{-t(\lambda^2 + \rho^2)}$ for $t \geq 0$, with inverse Jacobi transform being given by $k_t = h_t \star m^\vee$, where h_t is the heat kernel corresponding to $e^{t\mathcal{L}_{\alpha, \beta}}$ on \mathbb{R} . The functions m_t are rapidly decreasing and form an approximate identity, since the Jacobi heat semigroup is ultracontractive. This is an easy calculation: The Jacobi transform being a unitary map from $L^2(d\mu)$ to $L^2(d\nu)$, we conclude that

$$\begin{aligned} \|h_t\|_{L^2(d\mu)} &= \|\widehat{h_t}\|_{L^2(d\nu)} = \left(\frac{1}{2\pi} \int_0^\infty |e^{-t(\lambda^2 + \rho^2)}|^2 |\mathbf{c}(\lambda)|^{-2} d\lambda \right)^{\frac{1}{2}} \\ &= e^{-t\rho} \left(\frac{1}{2\pi} \int_0^\infty e^{-2t\lambda^2} |\mathbf{c}(\lambda)|^{-2} d\lambda \right)^{\frac{1}{2}} \lesssim e^{-t\rho} \left(\frac{1}{2\pi} \int_0^\infty e^{-2t\lambda^2} (1 + \lambda)^{2\alpha} d\lambda \right)^{\frac{1}{2}} \\ &\lesssim e^{-t\rho}. \end{aligned}$$

Moreover $\|M_t\|_{\mathcal{CO}_p(\mathbb{R})} = \|M\|_{\mathcal{CO}_p(\mathbb{R})}$ for all $t > 0$, so once Proposition 3.1 has been established for rapidly decreasing kernels, the inequality $\|\psi k_t\|_{\mathcal{CO}_p} \lesssim \|M_t\|_{\mathcal{CO}_p(\mathbb{R})} = \|M\|_{\mathcal{CO}_p(\mathbb{R})}$ holds for all $t > 0$ as well. But then $\sup_{t>0} \|\psi k_t\|_{\mathcal{CO}_p} \lesssim \|M\|_{\mathcal{CO}_p(\mathbb{R})}$, implying that $\|\psi k\|_{\mathcal{CO}_p} = \lim_{t \rightarrow 0^+} \|\psi k_t\|_{\mathcal{CO}_p} \lesssim \|M\|_{\mathcal{CO}_p(\mathbb{R})}$.

Proof of Proposition 3.1. We may assume by duality that $p \in (1, 2]$ and by Remark 3.2 that m is rapidly decreasing. By the inversion formula for the Jacobi transform, the kernel k may thus be written as $k(t) = \int_0^\infty m(\lambda)\varphi_\lambda(t) d\nu(\lambda)$ for $t \geq 0$. For the present proof it suffices to terminate the asymptotic expansion of φ_λ from [15, Lemma 3.1] after two terms (corresponding to the case $M = 1$): Write $\mathcal{J}_\alpha(\lambda) = (\lambda)^{-\alpha} J_\alpha(\lambda)$, where J_α is the usual second order Bessel function of order α . Then

$$\varphi_\lambda(t) = c_\alpha \frac{t^{\alpha + \frac{1}{2}}}{\sqrt{\Delta(t)}} (a_0(t)\mathcal{J}_\alpha(\lambda t) + a_1(t)t^2\mathcal{J}_{\alpha+1}(\lambda t) + E_2(\lambda t)),$$

¹For $\alpha > \frac{1}{2}$, $\alpha > \beta > -\frac{1}{2}$, and suitable λ there exist constants $R_0, R_1 \in (1, \sqrt{\frac{\pi}{2}})$ with $R_0^2 < R_1$ such that for every $M \in \mathbb{N}$ and every $t \in [0, R_0]$

$$(5) \quad \varphi_\lambda^{(\alpha, \beta)}(t) = \frac{2\Gamma(\alpha + 1)}{\Gamma(\alpha + \frac{1}{2})\Gamma(\frac{1}{2})} \frac{t^{\alpha + \frac{1}{2}}}{\sqrt{\Delta(t)}} \sum_{m=0}^\infty a_m(t)t^{2m}\mathcal{J}_{m+\alpha}(\lambda t)$$

$$(6) \quad \varphi_\lambda^{(\alpha, \beta)}(t) = \frac{2\Gamma(\alpha + 1)}{\Gamma(\alpha + \frac{1}{2})\Gamma(\frac{1}{2})} \frac{t^{\alpha + \frac{1}{2}}}{\sqrt{\Delta(t)}} \sum_{m=0}^M a_m(t)t^{2m}\mathcal{J}_{m+\alpha}(\lambda t) + E_{M+1}(\lambda t),$$

with good estimates on the error term E_{M+1} and the functions a_m .

where $a_0(t) \equiv 1$, $|a_1(t)| \lesssim R_1^{-(\alpha+\frac{1}{2})}$, $|E_2(\lambda t)| \lesssim t^4$ if $|\lambda t| \leq 1$, and $|E_2(\lambda t)| \lesssim t^4 |\lambda t|^{-(\alpha+2)}$ if $|\lambda t| \geq 1$. Correspondingly,

$$\begin{aligned} \psi(t)k(t) &= c_\alpha \frac{t^{\alpha+\frac{1}{2}}}{\sqrt{\Delta(t)}} \psi(t) \left(\int_0^\infty m(\lambda) \mathcal{J}_\alpha(\lambda t) d\nu(\lambda) + a_1(t) t^2 \int_0^\infty m(\lambda) \mathcal{J}_{\alpha+1}(\lambda t) d\nu(\lambda) \right. \\ &\quad \left. + \int_0^\infty m(\lambda) E_2(\lambda t) d\nu(\lambda) \right) =: I_1(t) + I_2(t) + I_3(t). \end{aligned}$$

We presently analyze the contribution $I_3(t)$ from the error term E_2 . First note that by the \mathbf{c} -function estimates from (2) and Lemma 2.1,

$$\begin{aligned} (7) \quad & \int_0^\infty \frac{t^{\alpha+\frac{1}{2}}}{|\sqrt{\Delta(t)}|} \psi(t) \left| \int_{\mathbb{R}} m(\lambda) E_2(\lambda t) d\nu(\lambda) \right| |\Delta'(t)|^2 dt \\ & \lesssim \|M\|_\infty \int_0^\infty \frac{t^{\alpha+\frac{1}{2}}}{|\sqrt{\Delta(t)}|} \psi(t) \left\{ \int_{|\lambda t| \leq 1} |E_2(\lambda t)| |\mathbf{c}(\lambda)|^{-1} d\lambda + \int_{|\lambda t| \geq 1} |E_2(\lambda t)| |\mathbf{c}(\lambda)|^{-1} d\lambda \right\} dt \\ & \lesssim \|M\|_\infty \int_0^\infty t^{\alpha+\frac{1}{2}} |\sqrt{\Delta(t)}|^{1/2} \psi(t) \left\{ \int_{|\lambda| \leq \frac{1}{t}} t^4 (1 + |\lambda|)^{\alpha+\frac{1}{2}} d\lambda \right. \\ & \quad \left. + \int_{|\lambda| \geq \frac{1}{t}} t^4 |\lambda t|^{-(\alpha+2)} |\lambda|^{\alpha+\frac{1}{2}} d\lambda \right\} dt. \end{aligned}$$

The integral $\int_{|\lambda| \leq \frac{1}{t}} t^4 (1 + |\lambda|)^{\alpha+\frac{1}{2}} d\lambda$ is finite since $\alpha > -\frac{1}{2}$, and one computes that

$$\int_{|\lambda| \geq \frac{1}{t}} t^4 (1 + |\lambda|)^{\alpha+\frac{1}{2}} d\lambda = 2t^{-(\alpha+\frac{1}{2})}.$$

Collecting powers of t in the above integral and using that ψ is compactly supported in a neighborhood around $t = 0$, we conclude that the quantity in the last line of (7) may be bounded by $C\|M\|_\infty$. It thus follows from Proposition 2.3 that $I_3 \in \mathcal{CO}_p$ for all $p \in (1, 2]$.

We must also investigate the contributions I_1 and I_2 , and to this end we consider the even functions defined on \mathbb{R}_+ by $b_0(t) = \psi(t) \frac{t^{\alpha+\frac{1}{2}}}{\sqrt{\Delta(t)}}$ and $b_1(t) = \psi(t) \frac{t^{\alpha+\frac{1}{2}}}{\sqrt{\Delta(t)}} t^2 a_1(t)$, $t > 0$, together with the functions

$$\begin{aligned} \Psi_j(t) &= b_j(t) \int_{\mathbb{R}} m(\lambda) \mathcal{J}_{\alpha+j}(\lambda t) d\nu(t), \quad j = 0, 1 \\ &= b_j(t) \left\{ \int_{J_t} m(\lambda) \mathcal{J}_{\alpha+j}(\lambda t) d\nu(\lambda) + \int_{\mathbb{R} \setminus J_t} m(\lambda) \mathcal{J}_{\alpha+j}(\lambda t) d\nu(\lambda) \right\} =: \Psi_j^0(t) + \Psi_j^\infty(t), \end{aligned}$$

where $J_t = (-\frac{1}{t}, \frac{1}{t})$. Observe that $I_j = \Psi_j$ for $j = 0, 1$. The point is that for $\lambda \in \mathbb{R} \setminus J_t$, say, we have $|\lambda t| \geq 1$ and may use improved estimates for the modified Bessel function $\mathcal{J}_{\alpha+j}(\lambda t)$ obtained in [16, Appendix A], closely resembling those used for the proof of [22, Theorem 2.1]. We wish to prove that Ψ_j belongs to \mathcal{CO}_p , with convolution operator-norm proportional with $\|M\|_{\mathcal{M}_p}$. To this end one observes that the local contributions $\Psi_j^0, j = 0, 1$ belong to $L^1(d\mu)$

with norm proportional with $\|M\|_\infty$, since

$$\begin{aligned}
\|\Psi_j^0\|_{L^1(d\mu)} &\leq \int_0^\infty \left| \psi(t) t^{\alpha+2j+\frac{1}{2}} a_j(t) \int_{J_t} M(\lambda) \mathcal{J}_{\alpha+j}(\lambda t) \mathbf{c}(\lambda)^{-1} d\lambda \right| \sqrt{\Delta(t)} dt \\
&\lesssim \|M\|_\infty \int_0^\infty \psi(t) t^{\alpha+2j+\frac{1}{2}} |a_j(t)| \int_{J_t} |\lambda|^{\alpha+\frac{1}{2}} d\lambda \sqrt{\Delta(t)} dt \\
&\lesssim \|M\|_\infty \int_0^\infty \psi(t) t^{2j-1} |a_j(t)| \sqrt{\Delta(t)} dt = \|M\|_\infty \int_0^{R_0} \psi(t) t^{2j-1} |a_j(t)| \sqrt{\Delta(t)} dt \\
&\lesssim \|M\|_\infty \int_0^{R_0} t^{2j-1} R_1^{-(\alpha+j+\frac{1}{2})} t^{\alpha+\frac{1}{2}} dt \simeq \|M\|_\infty \int_0^{R_0} t^{\alpha+2j-\frac{1}{2}} dt
\end{aligned}$$

which is indeed finite since $\alpha > \frac{1}{2} > 0$ and $j = 0, 1$.

The functions Ψ_j^0 therefore give rise to L^p -bounded convolution operators satisfying the required norm estimate, so we concentrate on the global part Ψ_j^∞ . According to the standard asymptotic expansion for Bessel functions, [23, p. 199, Formula 1], we write

$$\mathcal{J}_{\alpha+j}(s) \sim s^{-(\alpha+j+\frac{1}{2})} \left(\cos(s + \delta) - \beta_\alpha \frac{\sin(s + \delta)}{2s} + O(s^{-2}) \right), \quad 1 \leq s \leq \infty,$$

with $\beta_\alpha = \alpha(\alpha - 1)$ and $\delta = -\frac{\alpha+j}{2}\pi$, leading to the decomposition

$$\begin{aligned}
\Psi_j^\infty(t) &= b_j(t) t^{-\alpha-j-\frac{1}{2}} \int_{\mathbb{R} \setminus J_t} m(\lambda) \lambda^{-\alpha-j-\frac{1}{2}} \cos(\lambda t + \delta) d\nu(\lambda) \\
&\quad - \frac{\beta_\alpha}{2} b_j(t) t^{-\alpha-j-\frac{3}{2}} \int_{\mathbb{R} \setminus J_t} m(\lambda) \lambda^{-\alpha-j-\frac{3}{2}} \sin(\lambda t + \delta) d\nu(\lambda) \\
&\quad + b_j(t) t^{-\alpha-j-\frac{1}{2}} \int_{\mathbb{R} \setminus J_t} m(\lambda) \lambda^{-\alpha-j-\frac{1}{2}} e_j(\lambda t) d\nu(\lambda) \\
&=: k_{j,0}(t) + k_{j,1}(t) + E_j(t),
\end{aligned}$$

where $|e_j(\lambda t)| = O(|\lambda t|^{-2})$. Let us write $k_{j,0}$ and $k_{j,1}$ slightly more systematically as

$$k_{j,k}(t) = c_k b_j(t) t^{-\alpha-j-k-\frac{1}{2}} \int_{\mathbb{R} \setminus J_t} m(\lambda) \lambda^{-\alpha-j-k-\frac{1}{2}} h_k(\lambda t + \delta) d\nu(\lambda), \quad j, k = 0, 1,$$

where $c_0 = 1$, $c_1 = -\frac{\beta_\alpha}{2}$, $h_0(x) = \cos x$, and $h_1(x) = \sin x$. The error terms E_j are readily estimated:

$$\begin{aligned}
\|E_j\|_{L^1(d\mu)} &\lesssim \int_0^\infty \left| b_j(t) \int_{\mathbb{R} \setminus J_t} m(\lambda) |\lambda t|^{-\alpha-j-\frac{5}{2}} d\nu(\lambda) \right| \Delta(t) dt \\
&\lesssim \int_0^\infty |b_j(t)| t^{-\alpha-j-\frac{5}{2}} \left(\int_{\mathbb{R} \setminus J_t} M(\lambda) |\lambda|^{-\alpha-j-\frac{5}{2}} |\lambda|^{\alpha+\frac{1}{2}} d\lambda \right) \Delta(t) dt \\
&\lesssim \|M\|_\infty \int_0^\infty |b_j(t)| t^{-\alpha-j-\frac{5}{2}} \left(\int_{\mathbb{R} \setminus J_t} |\lambda|^{-\alpha-j-\frac{5}{2}} |\lambda|^{\alpha+\frac{1}{2}} d\lambda \right) \Delta(t) dt \\
&\lesssim \|M\|_\infty \int_0^{R_0} \psi(t) t^{\alpha+\frac{1}{2}+2j} |a_j(t)| t^{-\alpha-j-\frac{5}{2}} t^{\alpha+\frac{1}{2}} dt \quad \text{since } \int_{\mathbb{R} \setminus J_t} |\lambda|^{-j-2} d\lambda < \infty \\
&\simeq \|M\|_\infty \times \begin{cases} \int_0^{R_0} \psi(t) t^{\alpha-\frac{3}{2}} dt & \text{for } j = 0 \\ \int_0^{R_0} \psi(t) t^{\alpha-\frac{1}{2}} dt & \text{for } j = 1 \end{cases}.
\end{aligned}$$

We thus see that the natural assumption that α be strictly greater than $-\frac{1}{2}$ does not lead to the desired estimate for E_0 . *Imposing the stronger requirement that $\alpha > \frac{1}{2}$ certainly solves this issue.*

The piece $k_{1,1}$ is just as easily handled; indeed,

$$\begin{aligned} \|k_{1,1}\|_{L^1(d\mu)} &\lesssim \int_0^\infty \psi(t) t^{\alpha+\frac{5}{2}-\alpha-\frac{5}{2}} \left(\int_{\mathbb{R} \setminus J_t} M(\lambda) \lambda^{-\alpha-\frac{5}{2}} \lambda^{\alpha+\frac{1}{2}} d\lambda \right) \sqrt{\Delta(t)} dt \\ &\lesssim \|M\|_\infty \int_0^{R_0} \psi(t) t^{\alpha+\frac{1}{2}} \left(\int_{\mathbb{R} \setminus J_t} \lambda^{-2} d\lambda \right) dt \lesssim \|M\|_\infty \int_0^{R_0} \psi(t) t^{\alpha+\frac{1}{2}} dt \end{aligned}$$

so $k_{1,1}$ is μ -integrable with the correct norm estimate, thereby establishing the assertion of the Proposition in the case where $j + \gamma = 2$.

Assume $j + \gamma = 1$ and fix a smooth function Φ on \mathbb{R} with $0 \leq \Phi \leq 1$, $\Phi \equiv 1$ on $[-R_0^{-1}, R_0^{-1}]$, and $\Phi \equiv 0$ on $\mathbb{R} \setminus [-2R_0^{-1}, 2R_0^{-1}]$. Correspondingly, write $k_{j,k} = K_{j,k} + E_{j,k}$, where

$$\begin{aligned} K_{j,k}(t) &= c_k b_j(t) t^{-\alpha-j-k-\frac{1}{2}} \int_{\mathbb{R} \setminus J_t} m(\lambda) (1 - \Phi(\lambda)) \lambda^{-\alpha-j-k-\frac{1}{2}} h_k(\lambda t + \delta) d\nu(\lambda) \\ E_{j,k}(t) &= c_k b_j(t) t^{-\alpha-j-k-\frac{1}{2}} \int_{\mathbb{R} \setminus J_t} m(\lambda) \Phi(\lambda) \lambda^{-\alpha-j-k-\frac{1}{2}} h_k(\lambda t + \delta) d\nu(\lambda). \end{aligned}$$

First observe that $\|E_{j,k}\|_{L^1(d\mu)}$ is bounded by

$$\begin{aligned} \int_0^\infty |b_j(t)| t^{-\alpha-j-k-\frac{1}{2}} \left(\int_{\mathbb{R} \setminus J_t} |M(\lambda)| \Phi(\lambda) |\lambda|^{-\alpha-j-k-\frac{1}{2}} |\lambda|^{\alpha+\frac{1}{2}} |h_k(\lambda t + \delta)| d\lambda \right) \Delta(t) dt \\ \lesssim \|M\|_\infty \int_0^\infty |b_j(t)| t^{-\alpha-j-k-\frac{1}{2}} \left(\int_{\mathbb{R} \setminus J_t} \Phi(\lambda) |\lambda|^{-1} |h_k(\lambda t + \delta)| d\lambda \right) \Delta(t) dt. \end{aligned}$$

The integral in λ is convergent since Φ has support in the set $[-2R_0^{-1}, 2R_0^{-1}]$. The integral in t is estimated as above, leading to an upper estimate of the form $\|M\|_\infty \int_0^{R_0} \psi(t) t^{\alpha+j-k+\frac{1}{2}} dt$; this integral is finite since the power in t is strictly greater than -1 due to the assumption that $j + k = 1$. This proves the assertion for $E_{j,k}$ in the case where $j + k = 1$, but the pieces $K_{j,k}$ cannot be treated nearly as naively. The problem is that the λ -integrand will now involve $1 - \Phi(\lambda)$, which will grow towards the constant 1 as $\lambda \in \mathbb{R} \setminus J_t$ increases. If we were to naively bound the function h_k by one, the resulting integral would be divergent, so one must exploit the oscillatory nature of the integrand. The $K_{j,k}$ are still μ -integrable as functions in t , since m is rapidly decreasing, but this is not enough to guarantee the type of norm bound we are after.

Instead we use an idea from the proof of [16, Lemma 5.6]: We will show that $\Delta K_{j,\gamma}$ is an L^p -convolutor for the Euclidean Fourier transform on \mathbb{R} and then use the principle of transference to infer that $K_{j,k}$ is an L^p -convolutor for the Jacobi transform with a suitable estimate on its operator norm. Note in this regard that the convolution kernel in [10, Theorem 4.1, Corollary 4.11, Corollary 4.12] merely has to be μ -integrable. By the Hörmander–Mihlin multiplier theorem it therefore suffices to show that the function $t \mapsto \Delta(t) K_{j,k}(t)$ is smooth and bounded on $\mathbb{R} \setminus \{0\}$ and that $|t| |(\Delta K_{j,k})'(t)|$ is bounded on $\mathbb{R} \setminus \{0\}$. Due to the presence of the function ψ in the definition of $K_{j,k}$ we may assume that $|t| \leq R_0$. Now consider the truncated integrals

$$I_+^R(t) = \int_{1/t}^R m(\lambda) (1 - \Phi(\lambda)) \lambda^{-\alpha-\frac{3}{2}} h_k(\lambda t + \delta) |\mathbf{c}(\lambda)|^{-2} d\lambda, \quad R > 0,$$

with

$$(8) \quad \frac{dI_+^R}{dt} = -m\left(\frac{1}{t}\right)(1 - \Phi\left(\frac{1}{t}\right))t^{\alpha+\frac{3}{2}}h_k(1+\delta)|\mathbf{c}\left(\frac{1}{t}\right)|^{-2} \\ + \int_{1/t}^R m(\lambda)(1 - \Phi(\lambda))\lambda^{-\alpha-\frac{1}{2}}h'_k(\lambda t + \delta)|\mathbf{c}(\lambda)|^{-2} d\lambda.$$

The integral in (8) is obviously majorized by

$$\int_{1/t}^R |m(\lambda)|(1 - \Phi(\lambda))\lambda^{-\alpha-\frac{1}{2}}\lambda^{2\alpha+1} d\lambda \leq \int_{1/R_0}^R |m(\lambda)|\lambda^{\alpha+\frac{1}{2}} d\lambda,$$

which is finite and independent of t , since m is rapidly decreasing. The same holds for derivatives with respect to t of said integral. It follows that the function

$$t \mapsto \psi(t)t^{2j-1}a_j(t)\sqrt{\Delta(t)} \int_{1/t}^R m(\lambda)(1 - \Phi(\lambda))\lambda^{-\alpha-\frac{1}{2}}h'_k(\lambda t + \delta)|\mathbf{c}(\lambda)|^{-2} d\lambda$$

is smooth and bounded away from 0, since for small t , the factor $t^{2j-1}a_j(t)$ behaves roughly like $t^{2j-1}t^{\alpha+\frac{1}{2}} = t^{2j+\alpha-\frac{1}{2}}$ which does not blow up near 0 as long as $\alpha \geq \frac{1}{2}$. Analogously the function

$$t \mapsto \psi(t)t^{2j-1}a_j(t)\sqrt{\Delta(t)}m\left(\frac{1}{t}\right)(1 - \Phi\left(\frac{1}{t}\right))t^{\alpha+\frac{3}{2}}h_k(1+\delta)|\mathbf{c}\left(\frac{1}{t}\right)|^{-2}$$

behaves roughly as $\psi(t)t^{2j-1}t^{\alpha+\frac{1}{2}}m\left(\frac{1}{t}\right)(1 - \Phi\left(\frac{1}{t}\right))t^{\alpha+\frac{3}{2}}t^{-(2\alpha+1)} = \psi(t)m\left(\frac{1}{t}\right)(1 - \Phi\left(\frac{1}{t}\right))t^{2j}$, which also remains smooth and bounded away from 0. Since the exact same arguments hold for the analogously defined integrals I_-^R , we firstly conclude that $(\Delta K_{j,k})'$ is bounded and smooth away from zero, and secondly – by similar calculations – that $t \mapsto |t| |(\Delta K_{j,k})'(t)|$ is bounded as well. The assumptions in the Hörmander–Mihlin multiplier theorem are therefore fulfilled.

Finally suppose $j = \gamma = 0$ and consider the function $P_s : \lambda \mapsto (1 - \Phi(\lambda))|\lambda|^{-s}\mathbf{c}(\lambda)^{-1}$, $s \in \mathbb{R} \setminus \{0\}$. By the usual \mathbf{c} -function estimates P_s is seen to be (smooth and) bounded on $\mathbb{R} \setminus \{0\}$ if $s \geq \alpha + \frac{1}{2}$. Moreover

$$\left| \frac{d}{d\lambda} P_s(\lambda) \right| \lesssim |\lambda|^{-s-1}|\mathbf{c}(\lambda)|^{-1} + |\lambda|^{-s} \left| \frac{d}{d\lambda} \mathbf{c}(\lambda)^{-1} \right| \lesssim |\lambda|^{-s-1}|\lambda|^{\alpha+\frac{1}{2}} + |\lambda|^{-s}|\lambda|^{\alpha-\frac{1}{2}} \lesssim |\lambda|^{-s+\alpha-\frac{1}{2}},$$

according to Lemma 2.1, so $\lambda \mapsto |\lambda| |P'_s|$ is bounded on $\mathbb{R} \setminus \{0\}$ whenever $s \geq \alpha + \frac{1}{2}$. In other words (by the Hörmander–Mihlin theorem) P_s is an L^p -multiplier for the Fourier transform whenever $s \geq \alpha + \frac{1}{2}$.

It follows easily that $MP_{\alpha+\frac{1}{2}}$ is again an L^p -multiplier for the Euclidean Fourier transform: Let T_m denote (as in Definition 2.4) the operator associated with an Euclidean multiplier m , that is $T_m(f) = (\mathcal{F}^{-1}m) \star f$. Then

$$T_{MP_{\alpha+1/2}}f = (\mathcal{F}^{-1}(MP_{\alpha+1/2})) \star f = (T_M \circ T_{P_{\alpha+1/2}})f.$$

Fix a compactly supported function $\tilde{\psi}$ that is smooth away from 0 and observe that the function

$$\tilde{K}_{0,0}(t) := t^\alpha \tilde{\psi}(t) \int_{\mathbb{R}} m(\lambda)(1 - \Phi(\lambda))|\lambda|^{-\alpha-\frac{1}{2}}e^{-i\lambda t} d\nu(\lambda) = ct^\alpha \tilde{\psi}(t) \mathcal{F}(MP_{\alpha+\frac{1}{2}})(t)$$

defines a convolution operator that is bounded on $L^p(\mathbb{R})$ (the convolution now referring to the Euclidean structure), hence yields an Euclidean L^p -multiplier. Its norm as an element in $\mathcal{CO}_p(\mathbb{R})$ may now be estimated as in the third paragraph on page 168 in [11], to the effect that

$\|\widetilde{K}_{0,0}\|_{\mathcal{CO}_p(\mathbb{R})} \lesssim \|M\|_{\mathcal{M}_p(\mathbb{R})}$. This is indeed allowed since the computation is purely Euclidean (no reference to any Jacobi analysis). The use of the space $A_p(\mathbb{R})$ in the reference just quoted is therefore justified and may be repeated. This concludes the proof of the local part of the multiplier theorem. \square

The use of transference in the above proof is precipitated by the lack of an analogue of the Herz restriction principle that was used in the proof of the analogous result [11, Proposition 3.2]. The proof thereby attains a Clerc–Stein-like flavour.

4. GLOBAL ANALYSIS

We use the Harish-Chandra expansion

$$\varphi_\lambda(t) = \mathbf{c}(\lambda)e^{(i\lambda-\rho)t}\phi_\lambda(t) + \mathbf{c}(-\lambda)e^{(-i\lambda-\rho)t}\phi_{-\lambda}(t), \quad \phi_\lambda(t) := \sum_{k=0}^{\infty} \Gamma_k(\lambda)e^{-2kt}$$

of the Jacobi function φ_λ to analyze the global part of the kernel k , just as in [22, Section 3], [11], and [3].

Lemma 4.1 (Gangolli estimates). *Let D be either a compact subset of $\mathbb{C} \setminus (-i\mathbb{N})$ or a set of the form $D = \{\lambda = \xi + i\eta \in \mathbb{C} \mid \eta \geq -\varepsilon|\xi|\}$ for some $\varepsilon \geq 0$. There exist positive constants K, d such that*

$$(9) \quad |\Gamma_k(\lambda)| \leq K(1+k)^d \text{ for all } k \in \mathbb{Z}_+, \lambda \in D.$$

Proof. See [8, Lemma 7]. \square

It follows that the expansion for $\phi_\lambda(t)$ converges uniformly on sets of the form $\{(t, \lambda) \in [c, \infty) \times D\}$, where c is a positive constant. More precisely, if $\lambda \in D$, and $c > 0$ is fixed, we see that

$$\forall t \geq c: |\phi_\lambda(t)| \leq \sum_{k=0}^{\infty} K(1+k)^d e^{-2kt} \lesssim \sum_{k=0}^{\infty} (1+k)^d e^{-2ck} \lesssim 1,$$

that is, $\phi_\lambda(t)$ is bounded uniformly in $\lambda \in D$ for $t \geq c > 0$. We will take $c = R_0$ in later applications. Since $\lambda \mapsto \phi_\lambda(t)$ is analytic in a strip containing the real axis, it follows as in the proof of [19, Lemma 7] that derivatives of ϕ_λ in λ are bounded independently of λ as well.

Observe that $\lambda \mapsto \mathbf{c}(-\lambda)^{-1}\Gamma_k(\lambda)$ is analytic in the half plane $\{\lambda \in \mathbb{C} : \operatorname{Im} \lambda > -\rho\}$. The following result is an easy adaptation of [11, Lemma 3.3], the proof of which we include for completeness.

Lemma 4.2. *The boundary value $(\Gamma_k)_\rho$ belongs to $\mathcal{M}_p(\mathbb{R})$ for all $p \in (1, \infty)$, and there exist positive constants C, d such that $\|(\Gamma_k)_\rho\|_{\mathcal{M}_p(\mathbb{R})} \leq Ck^d$ for all $k \geq 1$.*

Proof. As in [11], we prove the lemma by means of the Mikhlin multiplier theorem on \mathbb{R} . To this end we need a good uniform bound on the derivatives of $(\Gamma_k)_\rho$. The aforementioned standard Gangolli estimates do not suffice, but it can be proved as in [11, Lemma 3.3] that

$$(10) \quad \sup_{\{|\operatorname{Im} \lambda| \leq |\operatorname{Re} \lambda|\}} |\Gamma_k(\lambda)| \lesssim k^d$$

for a suitable constant d . The reader will have no trouble in repeating the proof, using that the root multiplicities m_α and $m_{2\alpha}$ (symmetric space parameters) are replaced by $2(\alpha - \beta)$ and $2\alpha + 1$ (with α, β being Jacobi parameters), respectively.

Consider the region $U = \{z \in \mathbb{C} \mid |\operatorname{Im}(z - i\rho)| \leq |\operatorname{Re}(z - i\rho)|\}$, together with the circle $\gamma : t \mapsto \frac{|\lambda|}{\sqrt{2}}e^{it} + (\lambda + i\rho)$, $t \in [0, 2\pi]$, with center in $\lambda + i\rho$ and radius $\frac{1}{\sqrt{2}}|\lambda|$ (which is completely

contained in the inner of U). An application of the Cauchy Integral Formula together with the improved Gangolli estimates (10) yields the estimate

$$\begin{aligned} \left| \frac{d\Gamma_k}{d\lambda}(\lambda + i\rho) \right| &= \left| \frac{1}{2\pi i} \int_{\gamma} \frac{\Gamma_k(z)}{(z - (\lambda + i\rho))^2} dz \right| \\ &= \frac{1}{2\pi} \left| \int_0^{2\pi} \frac{\Gamma_k(\gamma(t))}{\left(\frac{|\lambda|}{\sqrt{2}} e^{it}\right)^2 \sqrt{2}} dt \right| \lesssim \frac{1}{|\lambda|} \int_0^{2\pi} |\Gamma_k(\gamma(t))| dt \lesssim \frac{k^d}{|\lambda|}. \end{aligned}$$

The classical Mikhlin–Hörmander multiplier theorem on \mathbb{R} finishes the proof. \square

Proposition 4.3. *Let m be an even function that is bounded and holomorphic on Ω_1 , and assume that M and M_ρ are both in $\mathcal{M}_p(\mathbb{R})$. Then $(1 - \psi)k$ is an L^p -multiplier for the Jacobi transform with multiplier norm dominated by the sum of the multiplier norms of M and M_ρ .*

Proof. We may assume without loss of generality that M and M_ρ are rapidly decreasing, cf. Remark 3.2. Let $K(t) = (1 - \psi(t))k(t)\Delta(t)$. We shall use the principle of transference ([10, Corollary 4.11, 4.12]) to infer that $(1 - \psi)k$ is an L^p -multiplier for the Jacobi transform whenever K is an L^p -multiplier for the Euclidean Fourier transform on \mathbb{R} . The strategy will be to insert the Harish-Chandra series for φ_λ in the definition of $k(t)$, use a series expansion for $\Delta(t)$, and then analyze the various pieces individually.

As for $\Delta(t)$, observe that

$$\begin{aligned} \Delta(t) &= (e^t - e^{-t})^{2\alpha+1} (e^t + e^{-t})^{2\beta+2} = e^{2\rho t} (1 - e^{-2t})^{2\alpha+1} (1 + e^{-2t})^{2\beta+1} \\ &= e^{2\rho t} \left\{ \sum_{j=0}^{[[\alpha]] + [[\beta]]} c_j e^{-2jt} \right\} (1 - e^{-2t})^{\langle \alpha \rangle} (1 + e^{-2t})^{\langle \beta \rangle} =: e^{2\rho t} \sum_{j=0}^{[[\alpha]] + [[\beta]]} c_j \delta_{\alpha, \beta}(t) e^{-2jt}, \end{aligned}$$

for suitable constants c_j . Here $[[\alpha]]$ and $\langle \alpha \rangle$ denote the integer and the decimal part of $2\alpha + 1$, respectively. Note that $[[\alpha]] + [[\beta]] = [[2\rho]]$.

Moreover (by the inversion formula for the even function m)

$$\begin{aligned} m^\vee(t) &= \int_{\mathbb{R}} m(\lambda) \mathbf{c}(-\lambda)^{-1} e^{(i\lambda - \rho)t} \phi_\lambda(t) d\lambda = e^{-\rho t} \int_{\mathbb{R}} M(\lambda) \phi_\lambda(t) e^{i\lambda t} d\lambda \\ &= e^{-\rho t} \sum_{k=0}^{\infty} e^{-2kt} \int_{\mathbb{R}} M(\lambda) \Gamma_k(\lambda) e^{i\lambda t} d\lambda, \end{aligned}$$

since the Harish-Chandra series converges uniformly in a suitable set of λ , implying the following expansion formula for $K(t)$:

$$\begin{aligned} (11) \quad K(t) &= (1 - \psi(t)) e^{\rho t} \delta_{\alpha, \beta}(t) \sum_{j=0}^{[[2\rho]]} c_j e^{-2jt} \sum_{k=0}^{\infty} \int_{\mathbb{R}} M(\lambda) \Gamma_k(\lambda) e^{i\lambda t} d\lambda \\ &= (1 - \psi(t)) e^{\rho t} \delta_{\alpha, \beta}(t) \sum_{\ell=0}^{\infty} e^{-2\ell t} \sum_{j=0}^{[[2\rho]]} c_j \int_{\mathbb{R}} M(\lambda) \Gamma_{\ell-j}(\lambda) e^{i\lambda t} d\lambda, \end{aligned}$$

where $\Gamma_k \equiv 0$ for $k < 0$ by convention (notice the index shift in the summation). Define $a_l^+(t) = (1 - \psi(t)) e^{-2lt} \delta_{\alpha, \beta}(t) 1_{[0, \infty)}(t)$ and $a_l^-(t) = (1 - \psi(t)) e^{2lt} \delta_{\alpha, \beta}(t) 1_{(-\infty, 0]}(t)$, both viewed as even functions on \mathbb{R} , and define (in analogy with [11]) functions

$$b_j^\pm(t) = \int_{\mathbb{R}} M(\lambda) \Gamma_j(\lambda) e^{\pm(i\lambda + \rho)t} d\lambda, t \in \mathbb{R}, j \in \mathbb{N}_0, \text{ and } K_{\ell, j}(t) = a_\ell^-(t) b_{\ell-j}^-(t) + a_\ell^+(t) b_{\ell-j}^+(t).$$

A quick calculation establishes that $K = \sum_{\ell=0}^{\infty} \sum_{j=0}^{[2\rho]} c_j K_{\ell,j}$, and we now proceed to examine the individual $K_{\ell,j}$. The technique will be to view the integral defining b_j^{\pm} as a path integral and then shift the contour of integration towards the upper edge of the strip Ω_1 . See Figure 1.

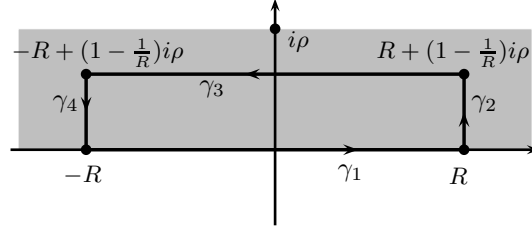


FIGURE 1. Change of contour-of-integration within (part of) Ω_1 (in gray)

Set $g_{\pm}(\lambda) = M(\lambda)\Gamma_k(\lambda)e^{\pm(i\lambda+\rho)t}$ for fixed $k \in \mathbb{N}_0$, $t > 0$, and parametrize the vertical segment γ_2 by $\gamma_2(s) = R + i\rho(1 - \frac{1}{R})s$, $s \in [0, 1]$. The g_{\pm} are holomorphic in Ω_1 and decrease rapidly as $|\operatorname{Re}\lambda| \rightarrow \infty$ with $\lambda \in \Omega_1$. Since

$$\begin{aligned} \int_{\gamma_2} g_{\pm}(\lambda) d\lambda &= i\rho(1 - \frac{1}{R}) \int_0^1 M\left(R + i\rho(1 - \frac{1}{R})s\right) \Gamma_k\left(R + i\rho(1 - \frac{1}{R})s\right) e^{\pm(i(R+i\rho(1-\frac{1}{R})s)+\rho)t} ds \\ &= i\rho(1 - \frac{1}{R}) e^{\pm(iR+\rho)t} \int_0^1 M\left(R + i\rho(1 - \frac{1}{R})s\right) \Gamma_k\left(R + i\rho(1 - \frac{1}{R})s\right) e^{\mp\rho(1-\frac{1}{R})st} ds \end{aligned}$$

with $|\operatorname{Im}(R + i\rho(1 - \frac{1}{R})s)| < |\rho|$ and $\operatorname{Re}(R + i\rho(1 - \frac{1}{R})s) = R$, it follows from the improved Gangolli estimates 4.2 that $|\Gamma_k(R + i\rho(1 - \frac{1}{R})s)| \lesssim k^d$ uniformly in R (as long as $R \geq |\rho|$). Hence

$$\left| \int_{\gamma_2} g_{\pm}(\lambda) d\lambda \right| \lesssim \int_0^1 |M(R + i\rho(1 - \frac{1}{R})s)| e^{\mp\rho(1-\frac{1}{R})st} dt \rightarrow 0 \text{ as } R \rightarrow \infty,$$

since $|M(z)|$ is rapidly decreasing in Ω_1 as $|\operatorname{Re}z| \rightarrow \infty$. An analogous investigation shows that also $|\int_{\gamma_4} g_{\pm}(\lambda) d\lambda| \rightarrow 0$ as $R \rightarrow \infty$.

Parametrize the horizontal segment γ_3 by $\gamma_3(s) = (1 - \frac{1}{R})i\rho - s$, $s \in [-R, R]$. Then

$$\begin{aligned} \int_{\gamma_3} g_+(\lambda) d\lambda &= - \int_{-R}^R M\left((1 - \frac{1}{R})i\rho - s\right) \Gamma_k\left((1 - \frac{1}{R})i\rho - s\right) e^{(i((1-\frac{1}{R})i\rho-s)+\rho)t} ds \\ &= - \int_{-R}^R M\left((1 - \frac{1}{R})i\rho - s\right) \Gamma_k\left((1 - \frac{1}{R})i\rho - s\right) e^{\frac{1}{R}\rho t} e^{-its} ds \\ &\rightarrow - \int_{\mathbb{R}} M(i\rho - s) \Gamma_k(i\rho - s) e^{-its} ds \text{ as } R \rightarrow \infty \\ &= -\mathcal{F}(H_k)(t), \end{aligned}$$

where $H_k(s) = M(i\rho - s)\Gamma_k(i\rho - s)$. Moreover

$$\begin{aligned}
\int_{\gamma_3} g_-(\lambda) d\lambda &= - \int_{-R}^R M\left((1 - \tfrac{1}{R})i\rho - s\right) \Gamma_k\left((1 - \tfrac{1}{R})i\rho - s\right) e^{-i((1 - \frac{1}{R})i\rho - s) + \rho)t} ds \\
&= - \int_{-R}^R M\left((1 - \tfrac{1}{R})i\rho - s\right) \Gamma_k\left((1 - \tfrac{1}{R})i\rho - s\right) e^{-\frac{1}{R}\rho t} e^{its} ds \\
&= - \int_{-R}^R M\left((1 - \tfrac{1}{R})i\rho + s\right) \Gamma_k\left((1 - \tfrac{1}{R})i\rho + s\right) e^{-\frac{1}{R}\rho t} e^{-its} ds \\
&\rightarrow - \int_{\mathbb{R}} (M\Gamma_k)_\rho(s) e^{-its} ds \text{ as } R \rightarrow \infty \\
&= -\mathcal{F}((M\Gamma_k)_\rho)(t).
\end{aligned}$$

In other words

$$(12) \quad K_{\ell,j}(t) = a_\ell^-(t) \mathcal{F}((M\Gamma_{\ell-j})_\rho)(t) + a_\ell^+(t) \mathcal{F}(H_{\ell-j})(t),$$

where $H_{\ell-j}(t) = M(i\rho - t)\Gamma_{\ell-j}(i\rho - t)$, just as on the bottom of [11, page 171].

Assuming $\ell > 0$, it follows as on page 172 in [11] that $\|a_\ell^-\|_{A_p(\mathbb{R})} \leq \|a_\ell^-\|_{A_2(\mathbb{R})} = \|\mathcal{F}(a_\ell^-)\|_{L^1(\mathbb{R})}$. Since the a_ℓ^- are compactly supported, the Sobolev embedding theorem implies the estimate

$$\|\mathcal{F}(a_\ell^-)\|_{L^1(\mathbb{R})} \lesssim \|a_\ell^-\|_{L^\infty(\mathbb{R})} \lesssim \|a_\ell^-\|_{L^2(\mathbb{R})} + \|(a_\ell^-)'\|_{L^2(\mathbb{R})}.$$

Note that by choice of ψ , $1 - \psi(t) \equiv 0$ for $|t| \leq R_0^{1/2}$ and $1 - \psi(t) \equiv 1$ for $|t| \geq R_0 > R_0^{1/2}$.

A favourable estimate for $\|a_\ell^-\|$ is obtained just as in [11] by direct calculation:

$$\begin{aligned}
\|a_\ell^-\|_{L^2(\mathbb{R})}^2 &= \int_{\mathbb{R}} (1 - \psi(t))^2 e^{4\ell t} 1_{(-\infty, 0]}(t) |\delta_{\alpha,\beta}(t)|^2 dt \\
&= \int_{-\infty}^0 (1 - \psi(t))^2 \delta_{\alpha,\beta}(t)^2 e^{4\ell t} dt = \int_{\text{supp}(1-\psi) \cap (-\infty, 0]} \delta_{\alpha,\beta}(t)^2 e^{4\ell t} dt \\
&\lesssim \int_{-\infty}^{-R_0} \delta_{\alpha,\beta}(t)^2 e^{4\ell t} dt \lesssim \int_{-\infty}^{-R_0} e^{4\ell t} dt \lesssim e^{-4\ell R_0}.
\end{aligned}$$

The estimate for $(a_\ell^-)'$ has no analogue in [11], [3] since the factor $\delta_{\alpha,\beta}$ is non-constant exactly when α, β are not half-integers. Its derivative must therefore be more carefully estimated. The issue is easily explained: as

$$\begin{aligned}
\delta'_{\alpha,\beta}(t) &= 2 \langle \alpha \rangle (1 - e^{-2t})^{\langle \alpha \rangle - 1} t e^{-2t} (1 + e^{-2t})^{\langle \beta \rangle} \\
&\quad - 2 \langle \beta \rangle (1 - e^{-2t})^{\langle \alpha \rangle} t e^{-2t} (1 + e^{-2t})^{\langle \beta \rangle - 1},
\end{aligned}$$

where $-1 \leq \langle \alpha \rangle - 1, \langle \beta \rangle - 1 < 0$, the function $(1 - \psi)\delta'_{\alpha,\beta}$ might not be in L^2 . But we have an additional exponential factor of $e^{2\ell t}$ in the definition of a_ℓ^- that will do the trick if we estimate more carefully. To this end introduce the auxiliary function $f(t) = e^{2\ell t} \delta_{\alpha,\beta}(t)$. Then

$$\begin{aligned}
f'(t) &= 2\ell e^{2\ell t} \delta_{\alpha,\beta}(t) + 2 \langle \alpha \rangle (1 - e^{-2t})^{\langle \alpha \rangle - 1} e^{2(\ell-1)t} (1 + e^{-2t})^{\langle \beta \rangle} \\
&\quad - 2 \langle \beta \rangle (1 - e^{-2t})^{\langle \alpha \rangle} e^{2(\ell-1)t} (1 + e^{-2t})^{\langle \beta \rangle - 1},
\end{aligned}$$

where $\|\ell e^{2\ell \cdot} (1 - \psi) \delta_{\alpha,\beta}\|_{L^2(\mathbb{R})}^2 \lesssim \ell^2 \int_{-\infty}^{-R_0} e^{4\ell t} dt \lesssim \ell e^{-4\ell R_0}$, which is still fine. The remaining two terms in the above expression for $f'(t)$ obviously satisfy the same type of L^2 -estimates, except possibly when $\ell = 1$, so let us assume $\ell = 1$. In this case $e^{2t} \delta'_{\alpha,\beta}(t)$ behaves roughly as $e^{-2t(\langle \alpha \rangle + \langle \beta \rangle - 1)}$ for $-\infty < t \ll -R_0 < 0$. Since $\langle \alpha \rangle + \langle \beta \rangle - 1 < 1$, it follows from the presence of the mitigating additional mitigating term $2e^{2t} \delta_{\alpha,\beta}(t)$ in the expression for $f'(t)$

that there exists some positive constant c such that $f'(t) \lesssim e^{-ct}$ for $-\infty < t \ll -R_0 < 0$, whence $\|(a_1^-)'\|_{L^2(\mathbb{R})}^2 \lesssim \int_{-\infty}^{-R_0} e^{-2ct} dt \lesssim e^{-2cR_0}$, with a similar bound for $\|(a_\ell^-)'\|_{L^2(\mathbb{R})}$ when $\ell \geq 2$. In conclusion it has thereby been shown that

$$\sum_{\ell=1}^{\infty} \|a_\ell^-\|_{W^{1,1}(\mathbb{R})} \leq \sum_{\ell=1}^{\infty} \left(\|a_\ell^-\|_{L^2(\mathbb{R})} + \|(a_\ell^-)'\|_{L^2(\mathbb{R})} \right) < \infty.$$

The considerations for a_ℓ^+ are similar so we shall not repeat the argument.

By Lemma 4.2 it now follows that $\|K_{\ell,j}\|_{\mathcal{CO}_p(\mathbb{R})} \lesssim \|a_\ell^-\|_{A_p(\mathbb{R})} \|(M\Gamma_{\ell-j})_\rho\|_{\mathcal{M}_p(\mathbb{R})}$, whence

$$\left\| \sum_{\ell=1}^{\infty} \sum_{j=0}^{[2\rho]} c_j K_{\ell,j} \right\| \leq \left(\sum_{\ell=1}^{\infty} \|a_\ell^-\|_{W^{1,1}(\mathbb{R})} \right) \|M_\rho\|_{\mathcal{M}_p(\mathbb{R})} \lesssim \|M_\rho\|_{\mathcal{M}_p(\mathbb{R})},$$

which is finite by assumption.

It remains to consider the case $\ell = 0$, in which case $j = 0$ as well. At this point we follow the argument on page 161 in [3] and introduce functions

$$\eta_\pm(t) = [(1 - \psi(t))1_{[0,\infty)}(\pm t) - 1]\delta_{\alpha,\beta}(t)e^{\mp\rho t};$$

then

$$\begin{aligned} K_{0,0}(t) &= a_0^-(t)b_0^-(t) + a_0^+(t)b_0^+(t) \\ &= (1 - \psi(t))\delta_{\alpha,\beta}(t)1_{[0,\infty)}(t)b_0^-(t) + (1 - \psi(t))\delta_{\alpha,\beta}(t)1_{(-\infty,0]}(t)b_0^+(t) \\ &= b_0^-(t) + (\eta_-(t) - 1)b_0^-(t) + b_0^+(t) + (\eta_+(t) - 1)b_0^+(t) \\ &= \mathcal{F}(M_\rho)(t) + \eta_-(t)\mathcal{F}(M) + \eta_+(t)\mathcal{F}(H)(t) + \mathcal{F}(H_\rho)(t), \end{aligned}$$

where $H(s) := M(-s)$. The first and last summands are precisely the kernels of the multipliers M_ρ and H_ρ , respectively. Obviously H_ρ is an L^p -multiplier since M_ρ is one by assumption and they have the same multiplier norm. Since $\mathcal{CO}_p(\mathbb{R})$ is an $A_p(\mathbb{R})$ -module we must now simply see that η_\pm belong to $A_p(\mathbb{R})$ but this is established by an analysis similar to the investigation of a_ℓ^\pm above. It thus follows that $\|K_{0,0}\|_{\mathcal{CO}_p(\mathbb{R})} \lesssim \|M_\rho\|_{\mathcal{M}_p(\mathbb{R})} + \|M\|_{\mathcal{M}_p(\mathbb{R})}$. \square

The change-of-contour technique was already used in the proof of [22, Proposition 4.5], see also [12, Proposition 5.1], although we have altered it slightly to take into account the nontangential boundary value along the upper edge. This point wasn't stressed in [11], [3].

There are other differences between the proof given above and the proofs of the analogous statements for rank one symmetric spaces ([11, Proposition 3.4]) and for Damek–Ricci spaces ([3, Proposition 4.5]). Most importantly we cannot use the Herz restriction principle since there are no subgroups to which multipliers are restricted. A more technical nuisance is in regards to $\Delta(t)$: The expansion of the function K into the pieces $K_{\ell,j}$ that was used in [11] and [3] ceases to be valid in the more general setting of Jacobi analysis, since α and β are no longer integers. It is insufficient to bound K pointwise by $|K(t)| \leq |H(t)|$ for a suitable convolutor H (where H is defined as K but by replacing Δ with $\sum_{j=0}^{[2\rho]+1} c_j e^{-2jt}$), our proof is somewhat more complicated. We thank the anonymous referee on a previous version of the paper for having pointed out this problem.

5. PROOF OF THE MULTIPLIER THEOREM

Proof. It suffices to prove that M and M_ρ belong to $\mathcal{M}_p(\mathbb{R})$, whenever m satisfies the hypotheses of Theorem 1.1, since the conclusion will then follow from Proposition 3.1 and Proposition 4.3.

To this end we proceed as in [11, p. 172–173] and use complex interpolation. The strategy is to ‘compress’ the strip Ω_1 and keep track of the nontangential boundary values of the modified multipliers along the edges of this compressed strip. More precisely, let $z \in \mathbb{C}$ with $\operatorname{Re} z \in [-1, 1]$ and denote by $(\omega m)_{z\rho}$ the nontangential boundary value of ωm along the upper edge of $\tilde{\Omega}_{z\rho} := \{\lambda \in \mathbb{C} : |\operatorname{Im} \lambda| < |\operatorname{Re} z|\rho\}$ if $\operatorname{Re} z > 0$ and the lower edge thereof if $\operatorname{Re} z < 0$. This is consistent with the previously defined nontangential boundary values $(\omega m)_\rho$, in the sense that $(\omega m)_{1,\rho}$ is what we previously denoted $(\omega m)_\rho$. Note that $(\omega m)_{\pm 1,\rho}$ belong to $\mathcal{M}_p(\mathbb{R})$; for $x = 1$ this is just the hypothesis, and for $x = -1$ this is due to the fact that ωm is even. Their respective kernels $T_{\pm 1} = \mathcal{F}^{-1}((\omega m)_{\pm\rho})$ are thus Euclidean L^p -convolutors. For $z \in \mathbb{C}$ with $\operatorname{Re} z \in [-1, 1]$ consider the tempered distribution T_z on \mathbb{R} that is given by $T_z = ((\omega m)_{z\rho})^\vee$, and use Euclidean convolution to define an operator S_z by $S_z f = T_z \star f$, $f \in \mathcal{S}(\mathbb{R})$. Clearly S_{1+iy} and S_{-1-iy} extend to bounded operators on $L^p(\mathbb{R})$ for all $y \in \mathbb{R}$, with operator norms $\|S_{\pm(1+iy)}\|_{L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R})} = \|S_{\pm 1}\|_{L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R})}$. Complex interpolation applied to the analytic family $\{S_z\}_{|\operatorname{Re} z| < 1}$ proves that S_z extends to a bounded operator on $L^p(\mathbb{R})$ for all $z \in (-1, 1)$, such that $\widehat{T_z} = (\omega m)_{z\rho}$ belongs to $\mathcal{M}_p(\mathbb{R})$ for the same range $z \in (-1, 1)$.

It thus remains to show that $m\mathbf{c}^{-1}$ and $(m\mathbf{c}^{-1})_\rho$ do belong to $\mathcal{M}_p(\mathbb{R})$. To this end we introduce the function $\mathbf{w} : \Omega_1 \rightarrow \mathbb{C}$, $\mathbf{w}(\lambda) = \omega(\lambda)^{-1}\mathbf{c}(\lambda)^{-1}$ and assert that \mathbf{w} and \mathbf{w}_ρ satisfy Hörmander type conditions on \mathbb{R} of arbitrarily high order, hence define Euclidean L^p -multipliers. From the identity $\mathbf{w}m\omega = m\mathbf{c}^{-1}$ we infer that $m\mathbf{c}^{-1}$ and $(m\mathbf{c}^{-1})_\rho$ are indeed L^p -multipliers, finishing the proof.

As for the Hörmander type estimates, note, for example, that

$$\left| \frac{d\mathbf{w}}{dx} \right| \leq \left| \frac{\mathbf{w}(x)}{\mathbf{w}(x)^2} \frac{d}{dx} (\mathbf{c}(x)^{-1}) \right| + \left| \frac{\mathbf{w}(x)'}{\mathbf{w}(x)^2} \mathbf{c}(x)^{-1} \right| \lesssim \frac{|x|^{\alpha+\frac{1}{2}+1}|x|^\alpha}{|x|^{2\alpha}} + \frac{|x|^{\alpha+\frac{1}{2}}|x|^{\alpha-1}}{|x|^{2\alpha}} \lesssim |x|^{-\frac{1}{2}}.$$

Additional derivatives in x will produce additional decay in $|x|$; we leave the elementary details to the reader. \square

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MATHEMATISCHES SEMINAR, CHRISTIAN-ALBRECHTS UNIVERSITÄT ZU KIEL, LUDEWIG-MEYN-STRASSE 4,
D-24098 KIEL, GERMANY

E-mail address: johansen@math.uni-kiel.de